Univariate continuous distributions: symmetries and transformations

M.C. Jones

Department of Mathematics & Statistics, The Open University, Walton Hall, Milton Keynes, MK7 6AA, U.K.

ABSTRACT

If the univariate random variable $X$ follows the distribution with distribution function $F$, then so does $Y = F^{-1}(1 - F(X))$. This known result defines the type of (generalised) symmetry of $F$, which is here referred to as T-symmetry; for example, ordinary symmetry about $\theta$ corresponds to $Y = 2\theta - X$. Some distributions, with density $f_S$, display a density-level symmetry of the form $f_S(x) = f_S(s(x))$, for some monotone transformation function $s(x) \neq x$; I call this S-symmetry. The main aim of this article is to introduce the S-symmetric dual of any (necessarily T-symmetric) $F$, and to explore the consequences thereof. Chief amongst these are the connections between the random variables following $F$ and $f_S$, and relationships between measures of ordinary symmetry based on quantiles and on density values.

Keywords: density-based asymmetry; probability integral transformation; quantile-based skewness; R-symmetry; S-symmetry.
1. Introduction

The usual notion of symmetry of a univariate absolutely continuous distribution is that the random variable \( X - \theta \) has the same distribution as \( \theta - X \) for some centre of symmetry \( \theta \). This will be called “ordinary symmetry” in this article.

Even when not ordinary symmetric, a distribution might have an alternative symmetry such as, for non-negative \( X \), log-symmetry, that is, the ordinary symmetry of the distribution of \( \log X \). This is equivalent to saying that \( X \) has the same distribution as \( \phi^2 / X \) for some \( \phi \). See Seshadri (1965). In fact, as is known, every distribution has a symmetry of this sort: for each distribution, there is a unique continuous decreasing function \( t \) such that \( X \) has the same distribution as \( t(X) \). Here, I shall call this ‘T-symmetry’. See Section 2.1.

Recently, there has been interest in an alternative, density-based, symmetry, the identification of a continuous monotone transformation function \( s(x) \neq x \) such that \( f(x) = f(s(x)) \) for all \( x \) in the support of the distribution with density \( f \). Here, I shall call this, as I have elsewhere, ‘S-symmetry’. See Section 2.2.

The main contribution of this article is to establish a duality between T- and S-symmetry. Specifically, in Section 3.1, I shall identify, for each distribution with a specific T-symmetry, the density of a corresponding distribution with S-symmetry \( \text{where the symmetry functions are the same, } s(x) = t(x) \text{ for all } x \). I shall also show how the random variables associated with these dual distributions can be obtained one from the other. After description of a number of special cases and antecedents in Section 4, the remaining main sections of the paper consider some further aspects of the interaction between T- and S-symmetry (Section 5) and some further aspects of T-symmetry alone (Section 6). There are brief closing remarks in Section 7.

2. Background: T- and S-Symmetries

2.1 T-Symmetry

Let the distribution of interest have invertible distribution function \( F \). To identify \( t \), start with the observation that if \( U \) is uniformly distributed on \((0, 1)\), then \( 1 - U \), and no other function of \( U \), is also distributed as \( U(0, 1) \). The probability integral transformation (PIT) then tells us that \( X = F^{-1}(U) \) follows the distribution \( F \). But because \( 1 - U \) is also uniform on \((0, 1)\), it must be the case that \( Y = F^{-1}(1 - U) \) follows \( F \) also. Thus, both \( X \) and \( Y = t(X) \), where

\[
t(x) = F^{-1}(1 - F(x)),
\]

have the same distribution, \( F \).

For an alternative derivation, consider determining \( t \) such that

\[
F(x) = P(X \leq x) = P(t(X) \leq x) = P(X \geq t^{-1}(x)) = 1 - F(t^{-1}(x))
\]
or equivalently \( F(t(x)) = 1 - F(x) \), and hence (1). This argument assumes that \( t \) is decreasing (as indeed is (1)); if \( t \) were increasing, \( F(x) = F(t^{-1}(x)) \) for all \( x \) implies \( t(x) = x \).

The beautiful observation (1), which applies to every continuous distribution \( F \), is certainly very far from new (e.g. Doksum, 1975, MacGillivray, 1986), but seems not to be very widely appreciated. I particularly wish to draw attention to the deep and insightful treatment of this and related issues by Kucerovsky, Marchand & Small (2005). See that paper for a comprehensive analysis additionally allowing transformations with singularities (which I shall touch on briefly in Section 6.1) and much more rigorous and wide-ranging mathematics than will be found here.

Here are some immediate general observations on (1). First, \( t(x) \) can be written in terms of the survivor function \( F(x) = 1 - F(x) \), trivially as \( t(x) = F^{-1}(F(x)) \) and equivalently as \( t(x) = F^{-1}(F(x)) \). The function \( t(x) \), as well as being decreasing, is self-inverse: \( t(t(x)) = x \); \( t^{-1}(x) = t(x) \). And, rearranging a formula above,

\[
F(t(x)) + F(x) = 1. 
\]

Of course, the distribution of \( X - t(X) \) is ordinary symmetric about zero.

The symmetry of this subsection, which to make distinct from the different symmetry to follow will be called ‘T-symmetry’, can be thought of as being on the level of distribution functions and hence intimately associated with transformations of random variables.

### 2.2 S-Symmetry

In contrast to the distribution-level \( T \)-symmetry of Section 2.1, in this subsection I assume the existence of a density function, supposed below to be differentiable except perhaps at its mode, and hence consider symmetry at the level of the density,

\[
f_S(s(x)) = f_S(x) 
\]

for all \( x \) and some function \( s(x) \neq x \). Motivated by the special case of R-symmetry introduced in the seminal work of Mudholkar & Wang (2007) which takes \( s(x) = \psi^2/x \) for \( x > 0 \), I call the property (3) ‘S–symmetry’ in Jones (2010, 2012). It is superficially reminiscent of the ‘generalized symmetry’ of Azzalini (2012) and Azzalini & Capitanio (2014, Section 1.3.2); however, a requirement of the latter that a certain determinant be unity affords no generalization over ordinary symmetry in the univariate case of interest here, but is designed for deployment as a multivariate extension.

Interest in the current paper centres on \( s \) being a one-to-one function. If \( s \) is decreasing, \( f_S \) has opposite signs at \( x \) and \( s(x) \) and hence \( f_S \) is unimodal or possibly, in the case of finite support, uniantimodal, with mode or antimode at \( x_0 \) such that \( s(x_0) = x_0 \). It also then follows that \( s \), like \( t \) in Section 2.1, must be self-inverse. Also
as for \( t, s(x) \neq x \) increasing is ruled out, in this case because it implies an increasing or decreasing density which would have to take equal values at \( x \) and \( s(x) \).

As noted in Chaubey, Mudholkar & Jones (2010), the R-symmetric distributions coincide with the Cauchy–Schlömilch distributions introduced by Baker (2008). Similarly, in the unimodal case, the S-symmetric distributions coincide with the ‘transformation of scale’ distributions of Jones (2010, 2014); see Section 4.4.

3. The S-Symmetric Dual of a Distribution

I will now use \( f_S \) specifically for the density of the S-symmetric dual of \( f \) defined in the theorem below, that is, the density of an S-symmetric distribution depending only on \( f \) and/or \( F \) and with

\[
s(x) = t(x) = F^{-1}(1 - F(x)).
\]  

**Theorem.** The density of the S-symmetric dual of the distribution \( F \) with density \( f \) is given by

\[
f_S(x) = \frac{2f(x)}{1 - t'(x)} = \frac{2f(x)f(t(x))}{f(x) + f(t(x))},
\]  

where \( t \) is given by (4), the latter representation being the harmonic mean of the density \( f(x) \) and the function \( f(t(x)) \).

**Proof.** The S-symmetry of \( f_S \) with \( s \) satisfying (4) is obvious from the final expression in (5). That \( f_S \) is a density follows from its nonnegativity and the fact that

\[
I \equiv \int f_S(x)dx = 2 \int \frac{f(x)f(t(x))}{f(x) + f(t(x))}dx = 2 \int \left[ f(x) - \frac{f^2(x)}{f(x) + f(t(x))} \right] dx.
\]

By using the substitution \( y = t(x) \) in the second part of the final integral, we get \( I = 2 - I \) so that \( I = 1 \). \( \square \)

I do not know how to construe and prove uniqueness of the construction above. However, I am not aware of any other candidates for this role.

When the integrals in the proof above have upper limit \( y \), we find that \( F_S(y) = 2F(y) - 1 + F_S(t(y)) \) where \( F_S \) is the distribution function of the dual S-symmetric distribution. Using (2) and rearranging leads to the following intriguing invariance relationship between probabilities of lying in certain intervals under \( f \) and \( f_S \):

\[
F_S(y) - F_S(t(y)) = F(y) - F(t(y)).
\]  

By differentiation of \( f_S \), any stationary point, \( x_p \), of it satisfies \( \ell(x_p) = \ell(t(x_p)) \) where \( \ell(y) = f'(y)/f^3(y) \). If \( f_S \) is unimodal, this means that its mode, \( x_0 \), must
equal the median, \( m_F \), of \( f \), because \( t(m_F) = F^{-1}(1 - F(m_F)) = m_F \); also, from (5), \( f_S(m_F) = f(m_F) \).

Now let \( X \sim f \), where \( \sim \) denotes ‘follows the distribution with density’, and \( X_S \sim f_S \) given by (5). It can readily be checked that \( X \) and \( X_S \) are related in the following way:

\[
X_S = \begin{cases} 
X & \text{with probability } \frac{1 - t'(X)}{1 - t(X)} = \frac{f(t(X))}{f(X) + f(t(X))}, \\
t(X) & \text{with probability } \frac{t'(X)}{1 - t(X)} = \frac{f(t(X))}{f(X) + f(t(X))},
\end{cases}
\] (7)

\[
X = \begin{cases} 
X_S & \text{with probability } 1/2, \\
t(X_S) & \text{with probability } 1/2.
\end{cases}
\] (8)

These relationships are reminiscent of Theorem 2.1 of Jones (2012) and for good reason: see Section 4.4 below. They can be used to good effect in random variate generation of S-symmetric distributions; for the main example thereof, see Section 4.2.

4. Special Cases and Antecedents

4.1 Ordinary Symmetry

If \( F \) is ordinary symmetric about \( \theta \), \( F(2\theta - x) = 1 - F(x) \) and so \( t(X) \), which has the same distribution as \( X \), is given by \( t(X) = F^{-1}\{F(2\theta - X)\} = 2\theta - X \), as expected. Also, since \( f(2\theta - x) = f(x) \), (5) yields \( f_S(x) = f(x) \), and there is no distinction between T- and S-symmetry.

4.2 R-Symmetry

If \( F \) on \( x > 0 \) is log-symmetric, \( F(\phi^2/x) = 1 - F(x) \) (Seshadri, 1965). By (2), the equidistribution here is of \( X \) and \( t(X) = \phi^2/X \). Using \( f(\phi^2/x) = x^2 f(x)/\phi^2 \), the S-symmetric, or more specifically R-symmetric (Mudholkar & Wang, 2007), dual of log-symmetric \( f \) has density

\[
f_R(x) = 2x^2 f(x)/(x^2 + \phi^2), \quad x > 0.
\] (9)

A particular example of this R-symmetric duality is when \( f \) is the density of \( 1/\sqrt{B} \) and \( B \) follows the Birnbaum-Saunders distribution, which is dual to the R-symmetric root-reciprocal inverse Gaussian, or CoGaussian, distribution (Mudholkar, Yu & Awadalla, 2014), that is, the distribution of \( 1/\sqrt{G} \) when \( G \) follows the inverse Gaussian distribution. In Jones (2012), I mention this and the consequential relationships between the Birnbaum-Saunders and inverse Gaussian distributions themselves, which give a derivation of the Michael, Schucany & Haas (1976) method for generating inverse Gaussian random variates.
The lognormal distribution, in its usual normal-based parametrisation, is both log-symmetric (about $e^{\mu}$) and R-symmetric (about $e^{\mu-\sigma^2}$). However, the lognormal distribution is not ‘self-dual’, and it is clear from (9) that no distribution can be.

4.3 Exponential and Power Law Symmetries

If $F$ is the exponential distribution with density $f(x) = \lambda e^{-\lambda x}$, $\lambda, x > 0$, then it is easy to show that $t(x) = -\log(1 - e^{-\lambda x})/\lambda \equiv t_e(x)$, so if $X$ has the exponential distribution, $t_e(X)$ has the same exponential distribution. Changing argument in (2), exponential T-symmetry corresponds to distributions $F$ on $x > 0$ such that $F(-\lambda \log u) + F(-\lambda \log(1 - u)) = 1$, $0 < u < 1$. The S-symmetric dual of the exponential distribution has density $2\lambda e^{-\lambda x}(1 - e^{-\lambda x})$, $\lambda, x > 0$.

This is a special case of the exponentiated exponential distribution (Gupta & Kundu, 1999). Use of the exponential symmetry represented by $t_e(x)$ in transformation of scale distributions is suggested from other considerations in Jones (2010, 2014).

If $F$ is the power law distribution with density $f(x) = \alpha x^{\alpha-1}$, $\alpha > 0$, $0 < x < 1$, then $t_p(X) \equiv (1 - X^{\alpha})^{1/\alpha}$ follows the same power law distribution. In this case, T-symmetry corresponds to distributions on $0 < x < 1$ such that $F(x^{\alpha}) + F((1 - x)^{\alpha}) = 1$. The S-symmetric dual of this $F$ has a more complicated density that is omitted.

4.4 Transforming Ordinary Symmetric Distributions

Write $F(x) = G(\Pi^{-1}(x))$ where $G$ is an arbitrary distribution ordinary symmetric about zero and $\Pi = F^{-1}(G)$ is the appropriate increasing function such that $\Pi(Y)$ follows the distribution $F$ when $Y$ comes from $G$ (with density $g$). Thus,

$$t(x) = \Pi \left( G^{-1} \left\{ 1 - G(\Pi^{-1}(x)) \right\} \right) = \Pi \left( G^{-1} \left\{ G(-\Pi^{-1}(x)) \right\} \right) = \Pi(-\Pi^{-1}(x)), \quad (10)$$

using the ordinary symmetry about zero of $G$. Now, since $f(x) = g(\Pi^{-1}(x))/\Pi'(\Pi^{-1}(x))$ and $t'(x) = -\Pi'(-\Pi^{-1}(x))/\Pi'(\Pi^{-1}(x))$, the density of the S-symmetric dual of $F$ is

$$2g(\Pi^{-1}(x)) / \Pi'(\Pi^{-1}(x)) + \Pi'(-\Pi^{-1}(x)).$$

The particularly simple form $f_S(x) = 2g(\Pi^{-1}(x))$ arises if $\Pi$ is chosen to satisfy

$$\Pi'(y) + \Pi'(-y) = 1 \quad \text{or essentially equivalently} \quad \Pi(y) - \Pi(-y) = y. \quad (11)$$
These are precisely the ‘transformation of scale’ distributions of Jones (2010, 2014). And they are dual to $F$ written as $G(\Pi^{-1}(x))$ using the same $G$ and $\Pi$.

In fact, because of the arbitrary nature of the choice of symmetric $g$, for any $t(x)$ given by (1), it is possible to choose $\Pi$ through a variation on the right-hand equation in (11), namely

$$\Pi^{-1}(x) = x - t(x),$$

to equate the class of S-symmetric distributions to the class of transformation of scale distributions. Since $\Pi^{-1} = G^{-1}(F)$, this is equivalent to specifying $G$ via the following ordinary symmetrisation of $F$:

$$G^{-1}(u) = F^{-1}(u) - F^{-1}(1 - u).$$

5. Further Aspects of T- and S-Symmetry Together

5.1 Measures of Ordinary Asymmetry

Ordinary asymmetry, that is, the degree to which a distribution departs from ordinary symmetry, might be measured by how far the symmetry transformation $F^{-1}(1 - F(x))$ departs from $-x$. Doksum’s (1975) ‘symmetry function’ is of precisely this form. It is $A_D(x) = \frac{1}{2}\{F^{-1}(1 - F(x)) + x\}$. Doksum argues that $A_D(x)$ should be compared with the natural centre of ordinary symmetry, the median $m_F$, yielding a functional asymmetry measure proportional to $F^{-1}(1 - F(x)) - 2m_F + x$. Further, making this quantity scale free by dividing by the corresponding scale measure $F^{-1}(1 - F(x)) - x$ yields the function

$$\gamma_F(x) \equiv \frac{F^{-1}(1 - F(x)) - 2m_F + x}{F^{-1}(1 - F(x)) - x}. \quad (12)$$

This differs from the more usual quantile-based asymmetry function of David & Johnson (1956),

$$\gamma_F(u) \equiv \frac{F^{-1}(1 - u) - 2m_F + F^{-1}(u)}{F^{-1}(1 - u) - F^{-1}(u)}, \quad 0 < u < 1, \quad (13)$$

only by the change of scale $u = F(x)$. See also MacGillivray (1986).

On the other hand, Critchley & Jones (2008) propose the following density-based asymmetry function for use with unimodal distributions (see also Boshnakov, 2007). This is of particular interest for unimodal S-symmetric distributions. Write $x_L(p)$ and $x_R(p)$, $0 < p < 1$, for the solutions of $f_S(x) = pf_S(x_0)$ to the left and right of the mode, $x_0$, respectively, when $f_S$ is unimodal; note that $x_L(p) = s(x_R(p))$. Then, their scaled asymmetry function takes the form

$$\gamma_{f_S}(p) \equiv \frac{x_L(p) - 2x_0 + x_R(p)}{x_L(p) - x_R(p)} = \frac{s(\gamma_{f_S}(p)) - 2x_0 + x_R(p)}{s(\gamma_{f_S}(p)) - x_R(p)}. \quad (14)$$
For the S-symmetric dual of $f$, $s(x) = F^{-1}(1 - F(x))$, $x_0 = m_F$, and if we set $x = x_R(p)$, the resulting density-based measure $\gamma_{fs}(x)$ obtained from (14) is the same as $\gamma_F(x)$ at (12). The versions of (12) based on a change of scale to (0, 1) differ only in the way this is done: distribution-based $u = F(x)$ in (13) and density-based $p = f_S(x)/f_S(m_F)$ in (14).

5.2 An Intermediate Distribution

With $t$ given by (4), from (7), $t(X_S)$ is like $X_S$ with the probabilities reversed:

$$t(X_S) = \begin{cases} X & \text{with probability } \frac{-F'(x)}{1-F(x)} = \frac{f(x)}{f(x)+f(t(x))}; \\ t(X) & \text{with probability } \frac{1}{1-F(x)} = \frac{f(x)}{f(x)+f(t(x))}. \end{cases}$$

(15)

The distribution of $t(X_S)$ has density

$$f_{T(S)}(x) = \frac{2f^2(x)}{f(x)+f(t(x))},$$

(16)

which features in the proof of the theorem in Section 3. As with (5), it is superficially surprising that this is a bona fide density.

6. Further Aspects of T-Symmetry Alone

6.1 Two Equi-Distributed Transformations?

The standard Cauchy distribution provides a special case in which $X$ has the same distribution as both $-X$ and $1/X$. Since $F(x) = (1/2) + (\tan^{-1} x)/\pi$, (1) gives $t(x) = -x$.

The equi-distribution of $X$ and $1/X$ arises by allowing transformations with singularities (for much more on the latter, see Kucerovsky, Marchand & Small, 2005). Equi-distribution of $X$ and $-X$ remains unique among continuous decreasing transformations. Another aspect of this is that $1/X$ is the unique continuous decreasing, and hence equi-distributed, transformation for the half-Cauchy distribution (see also Seshadri, 1965). This is the standard Cauchy distribution truncated at 0; it has $F(x) = 2(\tan^{-1} x)/\pi$, for which (1) gives $t(x) = \tan\{(\pi/2) - \tan^{-1} x\} = 1/x$.

6.2 Survival Copulas

Suppose that $X_1 \sim f_1$ with distribution function $F_1$ and $X_2 \sim f_2$ with distribution function $F_2$ are jointly distributed with distribution function $F(x_1, x_2)$. Of course, the joint distribution of $U_1 = F_1(X_1)$ and $U_2 = F_2(X_2)$ is the copula, $C$, associated with $F$. Now, a well known alternative copula associated with this copula is its survival copula, $\hat{C}(v_1, v_2) = v_1 + v_2 - 1 + C(1 - v_1, 1 - v_2)$, which is the distribution function.
of $1 - U_1$ and $1 - U_2$ (e.g. Nelsen, 2006, Section 2.6). The name, of course, arises because this copula is the distribution of $V_1 = F_1(X_1)$ and $V_2 = F_2(X_2)$ which, in the current context, can also be seen as $V_i = F_i(t_i(X_i))$ where $t_i(x) = F_i^{-1}(1 - F_i(x))$, $i = 1, 2$, as at (1).

The distribution function associated with the survival copula is simply $F(t_1(x), t_2(y))$. Its marginal distributions are $F_1$ and $F_2$, the same as those of $F$, but its dependence structure is that of $t_1(X_1)$ and $t_2(X_2)$ rather than $X_1$ and $X_2$.

6.3 Two Ways From $X$ to $Z$

Suppose that $X \sim f$ and $Z \sim \ell$ with distribution function $L$, say. Then the usual way of transforming $Z$ to get $X$, via the PIT, is $Z = L^{-1}\{F(X)\}$. This can equivalently be written $Z = L^{-1}\{F(X)\}$. However, again, since $F(X)$ and $1 - F(X)$ have the same distribution, this transformation is not, as might be assumed, unique: there is another pair of equivalent transformations with the same distribution as $Z$, namely, $L^{-1}\{F(X)\} = L^{-1}\{F(X)\}$. Of course, the latter pair are the result of applying the transformation $L^{-1}(L)$ to the former pair.

As just one minor example, the Weibull distribution arises as the distribution of $E^\beta$, say, where $E$ follows the exponential distribution with parameter $\lambda > 0$, and $\beta > 0$. What is not so well appreciated is that the Weibull distribution also arises as the distribution of $\{-\log(1 - e^{-\lambda E})/\lambda\}^\beta$.

7. Closing Remarks

To repeat, T-symmetry is a property of every univariate continuous distribution, and it and its consequences may well be familiar to many readers. S-symmetry, on the other hand, defines a particular class of distributions, those with the density symmetry property (3) for some function $s(x) \neq x$. The main aim of this article — the theorem of Section 3 — has been to introduce the S-symmetric analogue of any given (T-symmetric) distribution $F$ and to explore the consequences thereof, chief amongst which may be the random variable connections at the end of Section 3 and the relationships between measures of ordinary symmetry in Section 5.1.

References


